J Stat Phys (2010) 138: 2–7
DOI 10.1007/s10955-009-9912-0

Chaos Theory Yesterday, Today and Tomorrow

Y.G. Sinai

Received: 13 December 2009 / Accepted: 16 December 2009 / Published online: 5 January 2010

Abstract This paper gives a short historical survey of basic events which had happend during the developmentb of chaos theory.

Keywords Anosov systems · Bernoulli shift · Chaos · Entropy · Geodesic flow · K-system · Standard map

My personal experience shows that people in the West consider the development of Chaos Theory differently from their Russian colleagues, mathematicians and physicists. In this text, I will try to present both points of view. It was the great Russian mathematician A.N. Kolmogorov who stressed from the very beginning the importance of ergodic theory for the explanation of many complicated phenomena in nature. Because he had worked on problems of turbulence in the 1940s, he mainly had in mind fluid dynamics. However, he considered his works on scaling laws in hydro-dynamics purely phenomenological and did not anticipate that they would eventually have rigorous mathematical justification. The first seeds of chaos theory can be seen in his famous talk in Amsterdam during the International Mathematical Congress in 1954 (see [20]). During this Congress Kolmogorov only briefly met von Neumann but they didn't have the opportunity to discuss together many fundamental problems. From the paper by Dyson [16] one can have an impression that at that time von Neumann also had some ideas of chaos in connection with his studies of weather prediction.

Many people share the point of view that the beginning of chaos theory dates back to 1959 when the Kolmogorov's paper [21] on the entropy of dynamical system appeared. In that paper, Kolmogorov solved the famous problem in ergodic theory. Using the modern terminology one can say that he proved that Bernoulli shifts with different value of entropy cannot be metrically isomorphic. In particular, this was true for Bernoulli shifts with probabilities $(\frac{1}{2}, \frac{1}{2})$ and $(\frac{1}{3}, \frac{1}{3}, \frac{1}{3})$. The problem of defining isomorphism classes of these shifts

Y.G. Sinai (✉)
Mathematics Department, Princeton University, Princeton, NJ, USA
e-mail: sinai@math.princeton.edu

Y.G. Sinai
Landau Institute of Theoretical Physics, Moscow, Russia

appeared after the famous paper by von Neumann on spectral theory of dynamical systems (see [43]) and von Neumann himself stressed the importance of this problem. According to S. Kakutani (private communication), he personally discussed with von Neumann the possibility of using the ideas of thermodynamics for attacking this problem. However, there is no reliable evidence for this.

In 1959, Kolmogorov was also very much interested in ideas about the complexity of functional spaces which he used in his famous works on the Hilbert XIII-th problem. At that time Kolmogorov apparently believed that dynamical systems generated by deterministic equations of motion differ from dynamical systems arising in probability theory even from the metrical point of view. According to his idea entropy was the invariant that would allow to distinguish these two types of dynamics.

An even more dramatic part of the story follows. Kolmogorov prepared his text for publication in Doklady, submitted it and went to Paris where he spent the whole semester. His final text was substantially different from what he presented in his lecture course given earlier where he gave a complete and absolutely correct theory of entropy for Bernoulli shifts. However, in the text of his paper he introduced an important notion of quasi-regular dynamical systems and formulated the theorem according to which the entropy was an invariant of quasi-regular systems.

During Kolmogorov's absence, I was interested in proposing the definition of entropy which could be used for arbitrary dynamical systems. Since at that time I already understood how finite partitions generate stationary random processes and other general results like Macmillan theorem were already known, it was not so difficult. However, since the Kolmogorov paper already existed it was not clear whether another definition was needed and deserved the publication.

The situation changed after Rokhlin proposed an example in which Kolmogorov's definition of entropy did not provide an invariant of dynamical systems. At the same time, Rokhlin was aware of my definition of entropy and proposed calculating the entropy of the group automorphism of the two-dimensional torus which I did with the help of Kolmogorov and Rokhlin. At that time, there was no understanding that these automorphisms are quasi-regular in the sense of Kolmogorov. However, it became clear that my definition of entropy had some value and the paper with this definition was published (see [35]). Soon after, many examples of deterministic dynamical systems with positive entropy were discovered.

After these examples appeared, a question arose concerning those properties of dynamics which produce systems with positive entropy. It is hard to say who was the first to formulate explicitly that the instability of dynamics was exactly that property. Some hints that instability can generate random phenomena can be found in the textbook by Poincare on probability theory, in the works by Birkhoff and Hadamard, and in the book of Russian physicist Krylov (see [24]). It is most likely that at that time the words "deterministic chaos" appeared in the works of Chirikov, Ford, and Zaslavski and entropy theory became popular among physicists. Chirikov introduced the famous standard map and formulated the hypothesis that the standard map has positive entropy for large values of the parameter. This hypothesis stays open until now. The book by Lichtenberg and Lieberman (see [25]) was published at that time and added to the popularity of chaos.

One can also mention the paper by Lasota and Yorke "Period three implies chaos" which became attractive to non-experts. According to the authors, chaos is associated with the appearance of periodic orbits of all periods. Another notion which was also used in connection with the definition of chaos was "sensitive dependence on initial conditions."

The development of the mathematical theory went in two different directions. S. Smale was working on problems of topological classification of continuous dynamical systems.

Thom and Smale (see [39]) understood that in the multi-dimensional case this problem is different from its two-dimensional version. Smale proposed the concept of the so-called A-systems for which topological questions can be studied in complete detail. Anosov gave the general definition of unstable dynamical systems which are now called Anosov systems (see [2]). He proved the property of the so-called structure stability of Anosov systems, which states that small perturbations of Anosov systems are topologically equivalent to one another.

This result can be viewed as showing that the property of the system to have positive entropy is stable under small perturbations of the *rhs* of equations. Main examples of Anosov systems are group automorphisms of tori and geodesic flows on manifolds of negative curvatures. Many topological problems related to chaos theory can be found in the book by Palis and Takens (see [30]).

In ergodic theory, the central problem was the isomorphism problem for Bernoulli shifts and more general quasi-regular systems, which since that time were called K-systems (K stands for Kolmogorov, Krylov and quasi-regular systems). In the paper [36] it was shown that each ergodic system with positive entropy can be coded by Bernoulli shifts, but this coding is not one-to-one. A great progress was achieved in the series of remarkable papers by Ornstein and his co-authors (see [28]). He proved that any two Bernoulli shifts with the same value of entropy are isomorphic, and this was the end of the theory of Bernoulli shifts. Later, he proposed an example of two K-systems with the same value of entropy which are non-isomorphic. A simple example of this kind was given by Kalikov (see [19]).

In 1962, the famous American meteorologist E. Lorenz made a great discovery. Working on problems of convection, he proposed a simple non-linear system of three *ODE* and proved for it numerically and quantitatively that it had positive entropy (see [27]). After that there appeared several mathematical papers (Williams [44], Guckenheimer and Williams [17], Afraimovich, Bykov and Shilnikov [1]) that proposed a complete theory of Lorenz-like systems. Recently, Tucker [42] gave a computer-assisted proof that the Lorenz system for the values of parameters used by Lorenz actually had positive entropy. One can say that the Lorenz system is the simplest non-linear system beyond the class of linear systems which has positive entropy. Another example of this type is the so-called Roessler system.

After the discovery of Lorenz attractor entropy theory, the theory of deterministic chaos became very popular among physicists. French astrophysicist M. Henon was so excited by the Lorenz system that he proposed its discrete version which is now called the Henon map (see [18]). Benedicks and Carleson (see [5]) proved a deep theorem about chaotic properties of the Henon map. Several years before [5]-paper, Yakobson [45] proved a famous theorem about a family of one-dimensional maps $x \to ax(1-x)$ having an absolutely continuous invariant measure. The result by Benedicks and Carleson for the Henon map is analogous to Jakobson theorem.

New examples in the theory of chaos came from billiard theory. It was Krylov [24] who noticed the analogy between dynamical systems with elastic collisions and geodesic flows in spaces of negative curvature. The analogy was based on intrinsic instability which is common for both classes of systems.

Analysis of two-dimensional billiards with strictly concave boundaries appeared in [37]. Some time later Bunimovich (see [7]) proposed the Bunimovich stadium for which he proved that the dynamics are also purely chaotic. The boundary of the Bunimovich stadium consists of two semi-circles and two straight segments. The Bunimovich stadium is well-known among physicists, especially in connection with problems of quantum chaos. The book by Chernov and Markarian (see [11]) contains the theory of these types of billiards. Later Bunimovich proposed multi-dimensional versions of his stadium, the so-called

Bunimovich mushrooms (see [8, 9]). The first results for multi-dimensional billiards with a concave boundary can be found in the paper [12]. The Boltzmann model of gas which consists of a gas of hard balls was studied in [12] and in the works of Szasz, Kramli, Simanyi and other Hungarian mathematicians (see [3, 4, 10, 23, 34, 40]). One of the manifestations of chaos is the Central Limit Theorem which states that the distributions of properly normalized time averages converge to a Gaussian distribution in the limit $t \to \infty$. In the case of billiards this means that the probability distribution of the displacements of moving particles converge to the Wiener measure. One should stress that this probability distribution is induced by the distribution of the initial conditions only. The first results on CLT for billiards can be found in [10]. For the most recent progress, see the works by Szasz and his students [40, 41]. Lai-Sang Young studied the discrete maps generated by the dispersed billiards mentioned above, corresponding to two subsequent moments of collision. She proved that the binary correlation functions for this map decay exponentially (see [46, 47]). It was an unexpected result because it showed that the closeness of these billiards to processes of Markov type is much bigger than one could imagine. Dolgopyat (see [15]) proved the exponential decay of correlation for the three-dimensional Anosov flows. Later this result was extended by Liverani (see [26]).

In the general theory of deterministic chaos a large role was played by the so-called Lyapunov exponents. Oseledetz proved the existence of these exponents for general dynamical systems, having an invariant measure (see [29]). This laid the foundation to the famous Pesin Theory (see [31]). One of its main result was the theorem which stated that in the ergodic case, the entropy is the sum of positive Lyapunov exponents.

Also, at that time the so-called Markov partitions were introduced in [38], see an improved version in the book by Bowen [6]. In many cases, Markov partitions allow one to construct a simple coding of trajectories of chaotic systems. Using Markov partitions, one can define the so-called physical measures or SRB-measures. These are measures that appear as limits as $\underline{t \to \infty}$ of evolutions of initially smooth measures. In general, dynamical systems might not have an absolutely continuous invariant measure and these limits may be singular and be different for different directions of time. For the case of Anosov systems, these measures were constructed in [38] and for other systems, they were introduced in the works by Ruelle (see [32]) and Bowen [6]. Ruelle also found a beautiful property of SRB-measures. He proved that for almost all initial conditions with respect to any smooth initial measure time averages of nice functions converge as $\underline{t \to \infty}$ to the averages of these functions with respect to SRB-measures. In the works of Cohen and Gallavotti (see [14]) SRB-measures were used for the description of entropy production and other phenomena in non-equilibrium statistical mechanics. In paper [13] the so-called Gauss dynamics were considered. In this dynamics, the energy is preserved but the micro-canonical distribution is not invariant. This case gives an ideal example of singular SRB-measures.

In 1970, Ruelle and Takens published a paper entitled "On the nature of turbulence" (see [33]). There they introduced the notion of strange attractor which according to their point of view can be used for the description of onset of turbulence. Strange attractors were not found in equations of fluid dynamics but they were constructed in many other non-linear systems.

Looking back, one can see that the development of Chaos Theory went by bursts, as new ideas merged and new classes of dynamical systems could be studied. There are a number of glaring omissions in this text: Feigenbaum universality of period-doubling bifurcations and renormalization-group methods in dynamics, Arnold diffusion, the new progress in the theory of Interval Exchange Transformations and dynamics in Teichmuller spaces. It seems to be impossible to cover all the topics of the Chaos Theory in a short paper.

Nowadays, the main directions in chaos theory and ergodic theory are connected with the problem of unique ergodicity (see the recent survey by Sarnak on quantum unique ergodicity), number theory, Ratner theory, Margulis works on flows in homogeneous spaces and applications to number theory. All of these can be considered as part of the Chaos Theory today.

However, I believe that the future of the chaos theory will be connected with new phenomena in non-linear PDE and other infinite-dimensional dynamical systems, where we can encounter absolutely unexpected phenomena.

References

1. Afraimovich, V.S., Bykov, V.V., Shilnikov, L.P.: On attractive non-robust limiting sets of type of Lorenz attractor. Mosc. Math. Soc. **44**, 150–212 (1983)
2. Anosov, D.V.: Geodesic flows on closed Riemannian manifolds of negative curvature. Proc. Steklov Inst. **90**, 210 (1967)
3. Bálint, P., Chernov, N., Szasz, D., Toth, I.: Multi-dimensional semi-dispersing billiards: singularities and the fundamental theorem. Ann. Inst. Henri Poincare **3**, 451–482 (2002)
4. Bálint, P., Chernov, N., Szasz, D., Toth, I.: Geometry of multi-dimensional dispersing billiards. Astérisque **286**, 119–150 (2003)
5. Benedicks, M., Carleson, L.: The dynamics of the Henon map. Ann. Math. **133**(1), 73–170 (1991)
6. Bowen, R.: Equilibrium States and the Ergodic Theory of Anosov Diffeomorphisms. Lecture Notes in Math., vol. 470. Springer, Berlin (1975)
7. Bunimovich, L.A.: On the ergodic properties of nowhere dispersing billards. Commun. Math. Phys. **65**(3), 295–312 (1979)
8. Bunimovich, L.A.: Chaotic and non-chaotic mushrooms. Discrete Contin. Dyn. Syst. **22**(1–2), 63–74 (2008)
9. Bunimovich, L.A.: Chaotic billiards. Bull. Am. Math. Soc. **46**(4), 683–690 (2009)
10. Bunimovich, L., Chernov, N., Sinai, Ya.: Statistical properties of two-dimensional hyperbolic billiards. Russ. Math. Surv. **46**, 47–106 (1991)
11. Chernov, N., Markarian, R.: Chaotic Billards. Math. Surveys and Monographs, vol. 127. AMS, Providence (2006), 316 pp.
12. Chernov, N., Sinai, Ya.: Ergodic properties of some systems of two-dimensional disks and three-dimensional balls. Russ. Math. Surv. **42**(3), 153–174 (1987)
13. Chernov, N., Eyink, G., Lebowitz, J., Sinai, Ya.: Steady-state electrical conduction in the periodic Lorentz gas. Commun. Math. Phys. **154**(3), 569–601 (1993)
14. Cohen, E.G.D., Gallavotti, G.: Dynamical ensembles in nonequilibrium statistical mechanics. Phys. Rev. Lett. **74**, 2604–2697 (1995)
15. Dolgopyat, D.: On decay of correlations in Anosov flows. Ann Math. **147**(2), 357–309 (1998)
16. Dyson, F.: Birds and frogs. Not. Am. Math. Soc. **56**(2), 212–223 (2009)
17. Guckenheimer, J., Williams, R.F.: Structural stability of Lorenz attractors. Publ. Math. IHES **50**, 59–72 (1979)
18. Henon, M.: A two-dimensional mapping with a strange attractor. Commun. Math. Phys. **50**, 69–77 (1876)
19. Kalikov, S.: $T, T^{\{-1\}}$ transformation is not loosely Bernoulli. Ann. Math. **115**(2), 393–409 (1982)
20. Kolmogorov, A.N.: Théorie générale des systémes dynamique et mécanique classique. In: Proceedings of the International Congress of Mathematicians, vol. 1, pp. 315–1333. North-Holland, Amsterdam (1954)
21. Kolmogorov, A.N.: New metric invariant of transitive dynamical systems and automorphisms of lebesgue spaces. Dokl. Acad. Nauk SSSR **119**(5), 861–864 (1958)
22. Kolmogorov, A.N.: On the entropy per unit of time as a metric invariant of an automorphism. Dokl. Acad. Nauk SSSR **124**(4), 754–755 (1959)
23. Kramli, A., Simányi, N., Szasz, D.: The K-property of four billiard balls. Commun. Math. Phys. **144**(1), 107–148 (1992)
24. Krylov, N.S.: Works on the Foundations of Statistical Mechanics. Princeton University Press, Princeton (1979)
25. Lichtenberg, A.J., Lieberman, M.A.: Regular and Stochastic Motion. Springer, Berlin (1983)
26. Liverani, C.: On contact Anosov flows. Ann. Math. **159**(3), 1275–1312 (2004)
27. Lorenz, E.N.: Deterministic nonperiodic flow. J. Atmos. Sci. **20**, 130–141 (1963)

28. Ornstein, D.S.: Ergodic Theory, Randomness and Dynamical Systems. Yale University Press, New Haven (1994), 141 pp.
29. Oseledetz, V.: Multiplicative ergodic theorem, characteristics Lyapunov exponents of dynamical systems. Trans. Mosc. Math. Soc. **19**, 179–210 (1968)
30. Palis, J., Takens, F.: Hyperbolicity and Sensitive Chaotic Dynamics at Homoclinic Bifurcations. Cambridge Studies in Advanced Mathematics, vol. 35. Cambridge University Press, Cambridge (1993)
31. Pesin, Ya.: Characteristic Lyapunov exponents and smooth ergodic theory. Russ. Math. Surv. **32**(4), 55–111 (1977)
32. Ruelle, D.: Ergodic theory of differentiable dynamical systems. Publ. Math. IHES **50**, 27–58 (1979)
33. Ruelle, D., Takens, F.: On the nature of turbulence. Commun. Math. Phys. **20**(3), 167–192 (1971)
34. Simanyi, N.: Proof of the ergodic hypothesis for typical hard ball systems. Ann. Henri Poincare **5**(2), 203–233 (2004)
35. Sinai, Ya.: On the notion of entropy of dynamical system. Dokl. Acad. Nauk. SSSR **124**(4), 768–771 (1959)
36. Sinai, Ya.: Weak isomorphism of measure-preserving maps. Mat. Sb. **63**(1), 23–42 (1964)
37. Sinai, Ya.: Markov partitions and U-diffeomorphisms. Funct. Anal. Appl. **2**(1), 64–89 (1968)
38. Sinai, Ya.: Dynamical systems with elastic collisions. Russ. Math. Surv. **25**(2), 141–192 (1970)
39. Smale, S.: Differentiable dynamical systems. Bull. Am. Math. Soc. **73**, 747–817 (1967)
40. Szasz, D., Varjú, T.: Markov towers and stochastic properties of billiards. In: Modern Dynamic Systems and Applications, pp. 433–445. Cambridge University Press, Cambridge (2004)
41. Szasz, D., Varjú, T.: Local limit theorem for the Lorentz process and its recurrence in the plane. Ergod. Theory Dyn. Syst. **24**(1), 257–278 (2004)
42. Tucker, W.: The Lorenz attractor exists. C. R. Acad. Sci., Ser. I. Math. **328**(12), 1197–1202 (1999)
43. von Neumann, J.: Operatoren Methode in der Klassischen Mechanik. Ann. Math. **33**, 587–642 (1932)
44. Williams, R.F.: The structure of Lorenz attractors. Publ. Math. IHES **50**, 101–152 (1979)
45. Yakobson, M.V.: Absolutely continuous invariant measures for one-parameter families of one-dimensional maps. Commun. Math. Phys. **81**(1), 39–88 (1981)
46. Young, L.-S.: Statistical properties of dynamical systems with some hyperbolicity. Ann. Math. **147**, 585–650 (1998)
47. Young, L.-S.: Recurrent time and rate of mixing. Isr. J. Math. **110**, 153–188 (1999)